How to Compute Geoid Undulations (Geoid Height Relative to a Given Reference Ellipsoid) from Spherical Harmonic Coefficients for Satellite Altimetry Applications

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Contents

1 Introduction **1**

2 Prerequisites **2**

3 Geoid Height, Step by Step **2**

3.1 Preliminaries 2

3.2 Local ellipsoidal radius 3

3.3 Coordinates: geocentric versus geographic 3

3.4 Normal gravity 4

3.5 Permanent tide system 4

3.6 Reference ellipsoid 5

3.7 Fully normalized associated Legendre functions 6

3.8 Putting it all together 7

4 Nota Bene **8**

5 Notation **9**

1 Introduction

This document describes in a cookbook-fashion how to obtain geoid height at a given location on the Earth from spherical harmonic coefficients of a gravity model. The document is intended to aid "the uninitiated oceanographer" (Victor Zlotnicki, personal communication) in constructing geoid undulations relative to a given reference ellipsoid, defined by semi-major axis and flattening parameter, to be used with satellite altimetry products.

It is clearly beyond the scope of this cookbook to explain *how* certain formulas are developed from first principles and *why* the implicit approximations are appropriate. The interested reader is referred to standard textbooks such as Heiskanen and Moritz (1967) or Torge (1991)—where, incidently, all of the formulas stem from that are presented here.

2 Prerequisites

In order to use this cookbook you need the following information, usually provided with the data product (the spherical harmonic coefficients):

common symbol(s)	meaning
GM	value of the gravity-mass constant in m^3/s^2 for the reference ellipsoid
a	value of the semi-major axis of the reference ellipsoid in meters
GM_g, a_g	the same for the geopotential model (subscript g)
f	flattening parameter of the reference ellipsoid or equivalently
b	value of the semi-minor axis of the reference ellipsoid in meters
ω	angular velocity of the Earth's rotation in 1/s
$\bar{C}_{n,m}$, $\bar{S}_{n,m}$	the fully normalized spherical harmonic coefficients (or Stokes' coefficients) of the gravity model.

The parameters GM, a, f (or b), and ω define the reference ellipsoid. This reference ellipsoid for the geoid undulations must be exactly the same as the one used for the altimetry data product. If you do not have this information directly from the provider of the data product your geoid undulation will most likely be wrong, so make sure to ask the authors of the altimetry data to provide you with this information.

Often, the satellite altimetry product and the spherical harmonic coefficients refer to different *permanent tide systems*. You have to know which tide systems are used so that you can correctly convert between them (see below).

Furthermore, you need the geographic, or equally, the geodetic coordinates of the location(s) where you want to compute the geoid height. We will denote the geographic latitude by φ and the longitude by λ.

3 Geoid Height, Step by Step

3.1 Preliminaries

First, let's write down formulas for quantities that we will need lateron (Heiskanen and Moritz, 1967). First the relationship between Earth flattening parameter f, semi-major and semi-minor axes a and b:

$$f = \frac{a-b}{a} \quad \text{or} \tag{1}$$

$$b = a(1-f) \tag{2}$$

$$E = \sqrt{a^2 - b^2} \qquad \text{(linear eccentricity)} \tag{3}$$

$$e = \frac{E}{a} \qquad \text{(first numerical eccentricity)} \tag{4}$$

$$e' = \frac{E}{b} \qquad \text{(second numerical eccentricity)} \tag{5}$$

$$m = \frac{\omega^2 a^2 b}{GM} \qquad \text{(an abbreviation)} \tag{6}$$

$$\gamma_a = \frac{GM}{ab}\left(1 - \frac{3}{2}m - \frac{3}{14}e'm\right) \qquad \text{(gravity acc. at equator)} \tag{7}$$

$$\gamma_b = \frac{GM}{a^2}\left(1 + m + \frac{3}{7}e'm\right) \qquad \text{(gravity acc. at poles)} \tag{8}$$

$$x = \frac{a \cos\varphi \cos\lambda}{\sqrt{1 - e^2 \sin^2\varphi}} \qquad \text{(Cartesian coordinates} \tag{9}$$

$$y = \frac{a \cos\varphi \sin\lambda}{\sqrt{1 - e^2 \sin^2\varphi}} \qquad \text{relative to center} \tag{10}$$

$$z = \frac{a(1-e^2) \sin\varphi}{\sqrt{1 - e^2 \sin^2\varphi}} \qquad \text{of Earth)} \tag{11}$$

3.2 Local ellipsoidal radius

The local ellipsoidal radius r is computed as:

$$r(\varphi) = \sqrt{x^2 + y^2 + z^2} = a\sqrt{1 - \frac{e^2(1-e^2)\sin^2\varphi}{1 - e^2 \sin^2\varphi}}. \tag{12}$$

An approximation to this radius r is given by (Heiskanen and Moritz, 1967):

$$r(\varphi) = a(1 - f \sin^2\varphi). \tag{13}$$

3.3 Coordinates: geocentric versus geographic

Spherical harmonic models are formulated in geocentric (or ellipsoidal) coordinates as opposed to geographic (or geodetic) coordinates. The longitude is the same for both geographic and geocentric coordinates. But you have to convert your geographic (or geodetic) latitude φ into geocentric latitude $\bar{\varphi}$. The formula for this is (Torge, 1991)

$$\bar{\varphi} = \arctan \frac{z}{\sqrt{x^2 + y^2}} = \arctan \left[\left(\frac{b}{a} \right)^2 \tan \varphi \right]. \tag{14}$$

3.4 Normal gravity

The normal gravity γ_0 is a function of latitude. Compute γ_0 by this formula (Heiskanen and Moritz, 1967, p.76, eq. 2-109):

$$\gamma_0(\varphi) = \gamma_a \frac{1 + \kappa \sin^2 \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} \quad \text{with} \quad \kappa = \frac{b\gamma_b - a\gamma_a}{a\gamma_a} \tag{15}$$

3.5 Permanent tide system

Make sure that your spherical harmonic coefficients are in the right permanent tide system. Generally, satellite altimetry products use the mean-tide system. Conversion between different permanent tide systems involves either modifying one spherical harmonic coefficient or adding a zonally uniform correction to the geoid undulations. To convert zero-tide coefficients to mean-tide, use:

$$\begin{aligned} \bar{C}_{2,0}^{(\text{mean tide})} - \bar{C}_{2,0}^{(\text{zero tide})} &= 1 \times \frac{(-0.198\,\mathrm{m}) r^3 g}{a^2 GM \sqrt{5}} \\ &= -1.39 \times 10^{-8}, \end{aligned} \tag{16}$$

to convert tide-free coefficients to mean-tide, use

$$\begin{aligned} \bar{C}_{2,0}^{(\text{mean tide})} - \bar{C}_{2,0}^{(\text{tide free})} &= (1 + k) \times \frac{(-0.198\,\mathrm{m}) r^3 g}{a^2 GM \sqrt{5}} \\ &= (1 + k) \times (-1.39) \times 10^{-8}, \end{aligned} \tag{17}$$

where k s the (fundamentally unknowable) zero frequency Love number, which must be adopted. (For example, for EGM96, k=0.3 was adopted). g is the mean gravity.

Alternatively, you could add the permanent tide correction in the space domain, after computing the geoid height N (Rapp, 1989):

$$N^{(\text{mean tide})} - N^{(\text{zero tide})} = \\ 1 \times (-0.198\,\text{m}) \times \left(\frac{3}{2}\sin^2\bar{\varphi} - \frac{1}{2}\right) \quad \text{or} \tag{18}$$

$$N^{(\text{mean tide})} - N^{(\text{tide free})} = \\ (1+k) \times (-0.198\,\text{m}) \times \left(\frac{3}{2}\sin^2\bar{\varphi} - \frac{1}{2}\right). \tag{19}$$

This latter method appears to be the preferred one, because you can easily convert between permanent tide systems after the more expensive computation of the geoid undulation.

3.6 Reference ellipsoid

An important part of the computation is the subtraction of the reference ellipsoid, defined by the semi-major axis of the Earth a and the flattening parameter f. To do so, the zonal coefficients of the spherical harmonic gravity model $\bar{C}_{2n,0}$, $n = 1, 2, 3, 4, 5$ are corrected by

$$\bar{C}_{2n,0} \longrightarrow \bar{C}_{2n,0} + \frac{GM}{GM_\text{g}}\left(\frac{a}{a_\text{g}}\right)^n \cdot \frac{J_{2n}}{\sqrt{4n+1}}. \tag{20}$$

The factor $(GM/GM_\text{g}) \cdot (a/a_\text{g})^n$ accounts for the fact that neither the gravity mass constants nor the semi-major axes (just a scaling factor for the geopotential model) have to be the same for geopotential model and reference ellipsoid. In fact, very often they are *not* the same.[1] In general, it is sufficient to correct the zonal coefficients $\bar{C}_{2,0}$ through $\bar{C}_{10,0}$. The coefficients J_{2n} are determined from the parameters a, f, and the angular velocity ω of the Earth by an expansion (Heiskanen and Moritz, 1967, p.73, eq. 2-92):

$$J_{2n} = (-1)^{n+1}\frac{3(E/a)^{2n}}{(2n+1)(2n+3)}\left(1 - n + 5n\frac{C-A}{ME^2}\right). \tag{21}$$

This equation involves new parameters: the moments of inertia with respect to any axis in the equatorial plane A and with respect to the axis of rotation C (not to be confused with the spherical harmonic coefficients $\bar{C}_{n,m}$), the total mass M, and the

[1]The reason why this is important, is that you are substracting two potentials from each other, the potential of the model and the potential of the ellipsiod and in principles these two can be computed with different constants GM and a (see Smith, 1998).

linear eccentricity $E = \sqrt{a^2 - b^2}$. They can be computed by (Heiskanen and Moritz, 1967)

$$\frac{C-A}{ME^2} = \frac{1}{3}\left[1 - \frac{2}{15}\left(\frac{me'}{q_0}\right)\right] \tag{22}$$

$$m = \frac{\omega^2 a^2 b}{GM} \tag{23}$$

$$q_0 = \frac{1}{2}\left[\left(1 + \frac{3}{e'^2}\right)\arctan e' - \frac{3}{e'}\right]. \tag{24}$$

$e' = E/b$ is again the second numerical eccentricity. Subtracting the reference ellipsoid is a very sensitive operation, because two large numbers are subtracted from each other and this is always prone to errors. So if your coefficients J_{2n} are only slightly wrong, this will have a big effect on your solution. If you are lucky, the precomputed values of the coefficients J_{2n}, which are in principle defined by a, f, ω, and GM, are provided with the reference ellipsoid.

3.7 Fully normalized associated Legendre functions

For convenience, let us introduce the abbreviations $t = \sin\bar{\varphi}$ and $u = \cos\bar{\varphi}$. The fully normalized associated Legendre functions $\bar{P}_{mn}(t)$, sometimes also called fully normalized harmonics, can be computed from the conventional associated Legendre functions $P_{n,m}$ by (Torge, 1991):

$$\bar{P}_{n,m}(t) = \sqrt{k(2n+1)\frac{(n-m)!}{(n+m)!}}\, P_{n,m}(t), \qquad \text{with} \quad k = \begin{cases} 1 & \text{for } m = 0 \\ 2 & \text{for } m \neq 0. \end{cases} \tag{25}$$

The associated Legendre functions can be computed with the following recursive formulas (e.g., Abramowitz and Stegun, 1972, Bronstein and Semendjajew, 1991):

$$\begin{aligned} P_{n+1,0}(t) &= (2n+1)\, t\, P_{n,0}(t) - n\, P_{n-1,0}(t) \\ P_{n,n}(t) &= (2n-1)\, u\, P_{n-1,n-1}(t) \\ P_{n,m}(t) &= P_{n-2,m}(t) + (2n-1)\, u\, P_{n-1,m-1}(t) \end{aligned} \tag{26}$$

with the starting values

$$
\begin{aligned}
&P_{0,0}(t) = 1 \\
&P_{1,0}(t) = t, && P_{1,1}(t) = u \\
&P_{2,0}(t) = \frac{3}{2}\,t^2 - \frac{1}{2}, && P_{2,1}(t) = 3\,u\,t, \quad P_{2,2}(t) = 3\,u^2.
\end{aligned}
$$

However, these recursion formulas become numerically unstable for large n and m (> 120) and you may have to use other, more sophisticated formulas. These can be found in, for example, Paul (1978) and Holmes and Featherstone (2002). Here, we reproduce one method from Holmes and Featherstone (2002) for convenience:

For the fully normalized *non-sectorial* (i.e., $n > m$) $\bar{P}_{n,m}(t)$ you can use the following recursion:

$$
\bar{P}_{n,m}(t) = a_{n,m}\,t\,\bar{P}_{n-1,m}(t) - b_{n,m}\,\bar{P}_{n-2,m}(t), \quad \text{for all } n > m \tag{27}
$$

where

$$
\begin{aligned}
a_{n,m} &= \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}}, \\
b_{n,m} &= \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}}.
\end{aligned}
$$

The *sectorial* (i.e. $n = m$) $\bar{P}_{m,m}(t)$ serve as seed values for the recursion in (27). Starting from $\bar{P}_{0,0}(t) = 1$ and $\bar{P}_{1,1}(t) = \sqrt{3}\,u$, they can be computed from

$$
\bar{P}_{m,m}(t) = u\sqrt{\frac{2m+1}{2m}}\,\bar{P}_{m-1,m-1}(t), \quad \text{for all } m > 1, \tag{28}
$$

such that

$$
\bar{P}_{m,m}(t) = u^m\sqrt{3}\prod_{i=2}^{m}\sqrt{\frac{2i+1}{2i}}, \quad \text{for all } m > 1. \tag{29}
$$

3.8 Putting it all together

Now you are have all the quantities you need to compute the geoid undulation N as a function of geographical longitude λ and latitude φ by the well-known formula (Heiskanen and Moritz, 1967)

$$N(\lambda,\varphi) = \frac{GM_{\mathrm{g}}}{\gamma_0(\varphi)\, r(\varphi)} \sum_{n=2}^{L} \left(\frac{a_{\mathrm{g}}}{r(\varphi)}\right)^n \ldots \\ \sum_{m=0,n} \left[\bar{C}_{n,m}\cos(m\lambda) + \bar{S}_{n,m}\sin(m\lambda)\right] \bar{P}_{nm}(\sin\bar{\varphi}). \tag{30}$$

Here, $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are the spherical harmonic coefficients of degree n and order m. Note that in this formula, you need to use the mass gravity constant GM_{g} and the scale factor a_{g} of the geopotential model. $\bar{P}_{n,m}(\sin\bar{\varphi})$ are the fully normalized harmonics, or fully normalized associated Legendre functions. They are described in the previous section (Section3.7). Note that the harmonics $\bar{P}_{n,m}$ are evaluated at the geocentric latitude $\bar{\varphi}$, and *not* at the geographical latitude φ.

4 Nota Bene

We have neglected the time dependence of the geoid, in particular of the coefficient $\bar{C}_{2,0}$, but this dependence can easily incorporated into the procedure. The rate of change of $\bar{C}_{2,0}$ at the time that the gravity model refers to is usually provided with the model coefficients.

5 Notation

Symbol	Description	Units
GM_{g}	mass gravity constant of the geopotential model	$\mathrm{m}^3\ \mathrm{s}^{-2}$
a_{g}	semi-major axis of the geopotential model	meters
GM	mass gravity constant of the reference ellipsoid	$\mathrm{m}^3\ \mathrm{s}^{-2}$
a	semi-major axis of the reference ellipsoid	meters
f	flattening parameter of the reference ellipsoid	no units
b	semi-minor axis of the reference ellipsoid	meters
E	linear eccentricity	meters
e	first (numerical) eccentricity	no units
e'	second (numerical) eccentricity	no units
ω	angular velocity of the Earth's rotation	s^{-1}
λ	longitude	$^\circ E$ or radians
φ	geographic latitude	$^\circ N$ or radians
$\bar{\varphi}$	geocentric latitude	$^\circ N$ or radians
r	local elliptic radius	meters
g	mean gravity	$\mathrm{m\,s}^{-2}$
γ_0	local normal gravity	$\mathrm{m\,s}^{-2}$
γ_a, γ_b	normal gravity at the equator, at the poles	$\mathrm{m\,s}^{-2}$
$\bar{C}_{n,m}$	fully normalized spherical harmonic coefficients	no units
$\bar{S}_{n,m}$	(or Stokes' coefficients) of the gravity model	no units
$P_{n,m}$	associated Legendre functions of the first kind	no units
$\bar{P}_{n,m}$	fully normalized harmonics	no units

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